Order of Operations: It's not Just for PEMDAS Anymore [1]

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Proposal for Research Project on Classroom Learning

Early this semester in my MATH 1310: Calculus, Stochastics, and Modeling course, I gave a quiz on the Chain Rule. (The Chain Rule is a mathematical technique that facilitates computation of rates of change of complex functions, by relating these rates to those of the function's simpler constituents.) More than half of my class got off on the wrong foot by misinterpreting what the quiz was asking for: I requested that they calculate the rate of change of e(x3), but they thought I was asking for the rate of change of (ex)3. That is, they conflated the process of first cubing, and then exponentiating, with that of first exponentiating, and then cubing. They were thus doomed from the start. And I should have seen it coming: I have been thinking quite a bit lately about what I see as the fundamental issue behind this and many, many common mathematical errors and misperceptions.

The issue in question is that of (non-)commutativity, or in other words, of order of operations. (Two mathematical operations are said to commute if the order in which they are performed does not affect the final outcome.) In a nutshell, this issue amounts to the fact that, in mathematics (as elsewhere), "been there, done that" is often quite different from "done that, been there." Itmatters whether the doing comes before, or after, the being.

It matters whether you add numerators first, and then divide your results, as opposed to dividing first and then adding. Students in, for example, my MATH 1110/1120: The Spirit and Uses of Mathematics (a.k.a. "Math for Elementary Education") course sequence (and in many courses at various different levels) will often perform the two key operations in the reverse order; doing so amounts to the fundamental error many students make when adding fractions.

It matters whether you multiply functions together first, and then differentiate (that is, compute rates of change), as opposed to differentiating first and then multiplying. Students in, for example, MATH 1300: Calculus I will often perform the two key operations in the reverse order; doing so amounts to the fundamental misunderstanding so many student have of the "product rule."

It matters whether you differentiate (in the above sense) a signal first, and then compute the Fourier transform (that is, analyze the frequency content) of the result, as opposed to computing the Fourier transform and then differentiating. Students in, for example, MATH 4330/5330: Fourier Analysis will often perform the two key operations in the reverse order; doing so amounts to the fundamental misunderstanding so many student have of so many processes entailed by frequency (that is, Fourier) analysis.

While it is not surprising that there are real and great misconceptions underlying mathematical difficulties at all levels, it has been a fascinating discovery to me that so many of these difficulties have a common root!

I plan to develop thought-provoking, conceptual modules on order of operation/(non-) commutativity at various different levels, and to work these into the curricula of the various courses. By assessing conceptual understanding specifically (rather than focusing solely on “material” covered), near the beginnings and ends of the courses, I hope to gain insight into the “big picture” of what makes math (in Barbie’s estimation, and that of many others) so “hard.”

By studying the order-of-operations issue across a broad spectrum of math courses, I hope also to be able to see whether common misunderstandings from pre-calculus and below CONTINUE through calculus and beyond. (Are students carrying the same misconception along and applying it in familiar ways to unfamiliar topics, or are they engaging in an essentially NEW battle with (non-)commutavity each time they newly encounter it?)

Through my collaborations and affiliations with the School of Education, and my work on various interdisciplinary STEM education projects, I have a wide variety of resources available to help me with the design, implementation, and assessment of my proposed studies.

The line of inquiry proposed above has not yet, as far as I can tell, been systematically developed in the literature. I am very excited by the prospect of having the opportunity to pursue it.

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